21. An important problem in calculus is finding the area of a region. Sketch the parabola $y = 1 - x^2$ and shade in the region above the x-axis between $x = -1$ and $x = 1$. Then sketch in the following rectangles: (1) height $f(-\frac{3}{4})$ and width $\frac{1}{2}$ extending from $x = -1$ to $x = -\frac{1}{2}$. (2) height $f(-\frac{1}{4})$ and width $\frac{1}{2}$ extending from $x = -\frac{1}{2}$ to $x = 0$. (3) height $f(\frac{1}{4})$ and width $\frac{1}{2}$ extending from $x = 0$ to $x = \frac{1}{2}$. (4) height $f(\frac{3}{4})$ and width $\frac{1}{2}$ extending from $x = \frac{1}{2}$ to $x = 1$. Compute the sum of the areas of the rectangles. Based on your sketch, does this give you a good approximation of the area under the parabola?

22. To improve the approximation of exercise 21, divide the interval $[-1, 1]$ into 8 pieces and construct a rectangle of the appropriate height on each subinterval. Compared to the approximation in exercise 21, explain why you would expect this to be a better approximation of the actual area under the parabola.

23. Use a computer or calculator to compute an approximation of the area in exercise 21 using (a) 16 rectangles, (b) 32 rectangles, (c) 64 rectangles. Use these calculations to conjecture the exact value of the area under the parabola.

24. Use the technique of exercises 21–23 to estimate the area below $y = \sin x$ and above the x-axis between $x = 0$ and $x = \pi$.

25. Use the technique of exercises 21–23 to estimate the area below $y = x^3$ and above the x-axis between $x = 0$ and $x = 1$.

26. Use the technique of exercises 21–23 to estimate the area below $y = x^3$ and above the x-axis between $x = 0$ and $x = 2$.

EXPLORATORY EXERCISE

1. Several central concepts of calculus have been introduced in this section. An important aspect of our future development of calculus is to derive simple techniques for computing quantities such as slope and arc length. In this exercise, you will learn how to directly compute the slope of a curve at a point. Suppose you want the slope of $y = x^2$ at $x = 1$. You could start by computing slopes of secant lines connecting the point $(1, 1)$ with nearby points. Suppose the nearby point has x-coordinate $1 + h$, where h is a small (positive or negative) number. Explain why the corresponding y-coordinate is $(1 + h)^2$. Show that the slope of the secant line is $\dfrac{(1+h)^2 - 1}{1 + h - 1} = 2 + h$. As h gets closer and closer to 0, this slope better approximates the slope of the tangent line. Letting h approach 0, show that the slope of the tangent line equals 2. In a similar way, show that the slope of $y = x^2$ at $x = 2$ is 4 and find the slope of $y = x^2$ at $x = 3$. Based on your answers, conjecture a formula for the slope of $y = x^2$ at $x = a$, for any unspecified value of a.

1.2 THE CONCEPT OF LIMIT

In this section, we develop the notion of limit using some common language and illustrate the idea with some simple examples. The notion turns out to be a rather subtle one, easy to think of intuitively, but a bit harder to pin down in precise terms. We present the precise definition of limit in section 1.6. There, we carefully define limits in considerable detail. The more informal notion of limit that we introduce and work with here and in sections 1.3, 1.4 and 1.5 is adequate for most purposes.

As a start, consider the functions

$$f(x) = \frac{x^2 - 4}{x - 2} \quad \text{and} \quad g(x) = \frac{x^2 - 5}{x - 2}.$$

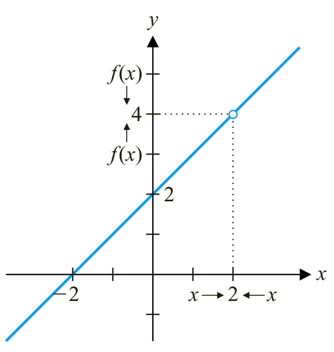


FIGURE 1.7a

$$y = \frac{x^2 - 4}{x - 2}$$

Notice that both functions are undefined at $x = 2$. So, what does this mean, beyond saying that you cannot substitute 2 for x? We often find important clues about the behavior of a function from a graph (see Figures 1.7a and 1.7b).

Notice that the graphs of these two functions look quite different in the vicinity of $x = 2$. Although we can't say anything about the value of these functions at $x = 2$ (since this is outside the domain of both functions), we can examine their behavior in the vicinity of

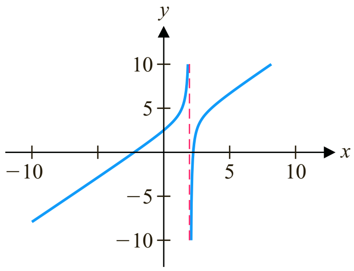


FIGURE 1.7b

$y = \frac{x^2 - 5}{x - 2}$

this point. We consider these functions one at a time. First, for $f(x) = \frac{x^2 - 4}{x - 2}$, we compute some values of the function for x close to 2, as in the following tables.

x	$f(x) = \frac{x^2-4}{x-2}$
1.9	3.9
1.99	3.99
1.999	3.999
1.9999	3.9999

x	$f(x) = \frac{x^2-4}{x-2}$
2.1	4.1
2.01	4.01
2.001	4.001
2.0001	4.0001

Notice that as you move down the first column of the table, the x-values get closer to 2, but are all less than 2. We use the notation $x \to 2^-$ to indicate that *x approaches 2 from the left side.* Notice that the table and the graph both suggest that as x gets closer and closer to 2 (with $x < 2$), $f(x)$ is getting closer and closer to 4. In view of this, we say that the **limit of $f(x)$ as x approaches 2 from the left** is 4, written

$$\lim_{x \to 2^-} f(x) = 4.$$

Likewise, we need to consider what happens to the function values for x close to 2 but larger than 2. Here, we use the notation $x \to 2^+$ to indicate that *x approaches 2 from the right side.* We compute some of these values in the second table.

Again, the table and graph both suggest that as x gets closer and closer to 2 (with $x > 2$), $f(x)$ is getting closer and closer to 4. In view of this, we say that the **limit of $f(x)$ as x approaches 2 from the right** is 4, written

$$\lim_{x \to 2^+} f(x) = 4.$$

We call $\lim_{x \to 2^-} f(x)$ and $\lim_{x \to 2^+} f(x)$ **one-sided limits.** Since the two one-sided limits of $f(x)$ are the same, we summarize our results by saying that the **limit of $f(x)$ as x approaches 2** is 4, written

$$\lim_{x \to 2} f(x) = 4.$$

The notion of limit as we have described it here is intended to communicate the behavior of a function *near* some point of interest, but not actually *at* that point. We finally observe that we can also determine this limit algebraically, as follows. Notice that since the expression in the numerator of $f(x) = \frac{x^2 - 4}{x - 2}$ factors, we can write

$$\begin{aligned} \lim_{x \to 2} f(x) &= \lim_{x \to 2} \frac{x^2 - 4}{x - 2} \\ &= \lim_{x \to 2} \frac{(x-2)(x+2)}{x - 2} && \text{Cancel the factors of } (x-2). \\ &= \lim_{x \to 2} (x + 2) = 4, && \text{As } x \text{ approaches 2, } (x+2) \text{ approaches 4.} \end{aligned}$$

where we can cancel the factors of $(x - 2)$ since in the limit as $x \to 2$, x is *close* to 2, but $x \neq 2$, so that $x - 2 \neq 0$.

x	$g(x) = \dfrac{x^2 - 5}{x - 2}$
1.9	13.9
1.99	103.99
1.999	1003.999
1.9999	10,003.9999

x	$g(x) = \dfrac{x^2 - 5}{x - 2}$
2.1	−5.9
2.01	−95.99
2.001	−995.999
2.0001	−9995.9999

Similarly, we consider one-sided limits for $g(x) = \dfrac{x^2 - 5}{x - 2}$, as $x \to 2$. Based on the graph in Figure 1.7b and the table of approximate function values shown in the margin, observe that as x gets closer and closer to 2 (with $x < 2$), $g(x)$ increases without bound. Since there is no number that $g(x)$ is approaching, we say that the *limit of g(x) as x approaches 2 from the left does not exist,* written

$$\lim_{x \to 2^-} g(x) \textit{ does not exist.}$$

Similarly, the graph and the table of function values for $x > 2$ (shown in the margin) suggest that $g(x)$ decreases without bound as x approaches 2 from the right. Since there is no number that $g(x)$ is approaching, we say that

$$\lim_{x \to 2^+} g(x) \textit{ does not exist.}$$

Finally, since there is no common value for the one-sided limits of $g(x)$ (in fact, neither limit exists), we say that the *limit of g(x) as x approaches 2 does not exist,* written

$$\lim_{x \to 2} g(x) \textit{ does not exist.}$$

Before moving on, we should summarize what we have said about limits.

A limit exists if and only if both corresponding one-sided limits exist and are equal. That is,

$$\lim_{x \to a} f(x) = L, \text{ for some number } L, \text{ if and only if } \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L.$$

In other words, we say that $\lim_{x \to a} f(x) = L$ if we can make $f(x)$ as close as we might like to L, by making x sufficiently close to a (on either side of a), but not equal to a.

Note that we can think about limits from a purely graphical viewpoint, as in example 2.1.

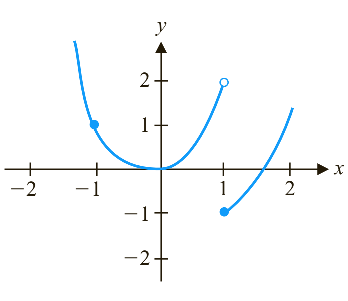


FIGURE 1.8
$y = f(x)$

EXAMPLE 2.1 Determining Limits Graphically

Use the graph in Figure 1.8 to determine $\lim_{x \to 1^-} f(x)$, $\lim_{x \to 1^+} f(x)$, $\lim_{x \to 1} f(x)$ and $\lim_{x \to -1} f(x)$.

Solution For $\lim_{x \to 1^-} f(x)$, we consider the y-values as x gets closer to 1, with $x < 1$. That is, we follow the graph toward $x = 1$ *from the left* ($x < 1$). Observe that the graph dead-ends into the open circle at the point (1, 2). Therefore, we say that $\lim_{x \to 1^-} f(x) = 2$. For $\lim_{x \to 1^+} f(x)$, we follow the graph toward $x = 1$ *from the right* ($x > 1$). In this case, the graph dead-ends into the solid circle located at the point (1, −1). For this reason, we say that $\lim_{x \to 1^+} f(x) = -1$. Because $\lim_{x \to 1^-} f(x) \neq \lim_{x \to 1^+} f(x)$, we say that $\lim_{x \to 1} f(x)$ does not exist. Finally, we have that $\lim_{x \to -1} f(x) = 1$, since the graph approaches a y-value of 1 as x approaches −1 both from the left and from the right. ■

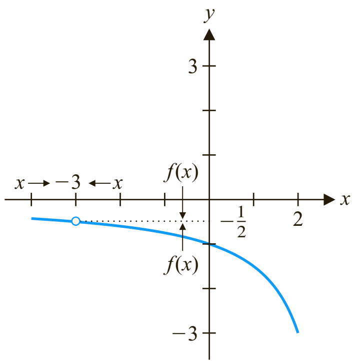


FIGURE 1.9

$$\lim_{x\to -3}\frac{3x+9}{x^2-9}=-\frac{1}{2}$$

x	$\frac{3x+9}{x^2-9}$
−3.1	−0.491803
−3.01	−0.499168
−3.001	−0.499917
−3.0001	−0.499992

x	$\frac{3x+9}{x^2-9}$
−2.9	−0.508475
−2.99	−0.500835
−2.999	−0.500083
−2.9999	−0.500008

EXAMPLE 2.2 A Limit Where Two Factors Cancel

Evaluate $\displaystyle\lim_{x\to -3}\frac{3x+9}{x^2-9}$.

Solution We examine a graph (see Figure 1.9) and compute some function values for x near -3. Based on this numerical and graphical evidence, it's reasonable to conjecture that

$$\lim_{x\to -3^+}\frac{3x+9}{x^2-9}=\lim_{x\to -3^-}\frac{3x+9}{x^2-9}=-\frac{1}{2}.$$

Further, note that

$$\lim_{x\to -3^-}\frac{3x+9}{x^2-9}=\lim_{x\to -3^-}\frac{3(x+3)}{(x+3)(x-3)} \quad \text{Cancel factors of } (x+3).$$

$$=\lim_{x\to -3^-}\frac{3}{x-3}=-\frac{1}{2},$$

since $(x-3)\to -6$ as $x\to -3$. Again, the cancellation of the factors of $(x+3)$ is valid since in the limit as $x\to -3$, x is *close* to -3, but $x\neq -3$, so that $x+3\neq 0$. Likewise,

$$\lim_{x\to -3^+}\frac{3x+9}{x^2-9}=-\frac{1}{2}.$$

Finally, since the function approaches the *same* value as $x\to -3$ both from the right and from the left (i.e., the one-sided limits are equal), we write

$$\lim_{x\to -3}\frac{3x+9}{x^2-9}=-\frac{1}{2}.$$ ■

In example 2.2, the limit exists because both one-sided limits exist and are equal. In example 2.3, neither one-sided limit exists.

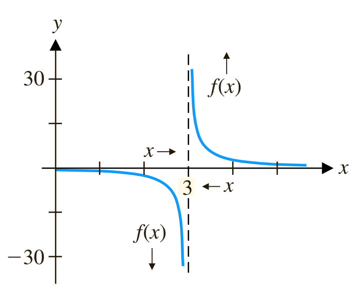


FIGURE 1.10

$$y=\frac{3x+9}{x^2-9}$$

x	$\frac{3x+9}{x^2-9}$
3.1	30
3.01	300
3.001	3000
3.0001	30,000

EXAMPLE 2.3 A Limit That Does Not Exist

Determine whether $\displaystyle\lim_{x\to 3}\frac{3x+9}{x^2-9}$ exists.

Solution We first draw a graph (see Figure 1.10) and compute some function values for x close to 3.

Based on this numerical and graphical evidence, it appears that, as $x\to 3^+$, $\frac{3x+9}{x^2-9}$ is increasing without bound. Thus,

$$\lim_{x\to 3^+}\frac{3x+9}{x^2-9} \text{ does not exist.}$$

Similarly, from the graph and the table of values for $x<3$, we can say that

$$\lim_{x\to 3^-}\frac{3x+9}{x^2-9} \text{ does not exist.}$$

Since neither one-sided limit exists, we say

$$\lim_{x\to 3}\frac{3x+9}{x^2-9} \text{ does not exist.}$$

Here, we considered both one-sided limits for the sake of completeness. Of course, you should keep in mind that if *either* one-sided limit fails to exist, then the limit does not exist. ■

x	$\frac{3x+9}{x^2-9}$
2.9	−30
2.99	−300
2.999	−3000
2.9999	−30,000

Many limits cannot be resolved using algebraic methods. In these cases, we can approximate the limit using graphical and numerical evidence, as we see in example 2.4.

EXAMPLE 2.4 Approximating the Value of a Limit

Evaluate $\lim_{x\to 0} \frac{\sin x}{x}$.

Solution Unlike some of the limits considered previously, there is no algebra that will simplify this expression. However, we can still draw a graph (see Figure 1.11) and compute some function values.

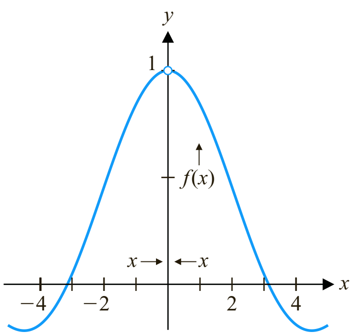


FIGURE 1.11

$\lim_{x\to 0} \frac{\sin x}{x} = 1$

x	$\frac{\sin x}{x}$
0.1	0.998334
0.01	0.999983
0.001	0.99999983
0.0001	0.9999999983
0.00001	0.999999999983

x	$\frac{\sin x}{x}$
−0.1	0.998334
−0.01	0.999983
−0.001	0.99999983
−0.0001	0.9999999983
−0.00001	0.999999999983

The graph and the tables of values lead us to the conjectures:

$$\lim_{x\to 0^+} \frac{\sin x}{x} = 1 \quad \text{and} \quad \lim_{x\to 0^-} \frac{\sin x}{x} = 1,$$

from which we conjecture that

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

In Chapter 2, we examine these limits with greater care (and prove that these conjectures are correct). ■

REMARK 2.1

Computer or calculator computation of limits is unreliable. We use graphs and tables of values only as (strong) evidence pointing to what a plausible answer might be. To be certain, we need to obtain careful verification of our conjectures. We see how to do this in sections 1.3–1.7.

EXAMPLE 2.5 A Case Where One-Sided Limits Disagree

Evaluate $\lim_{x\to 0} \frac{x}{|x|}$.

Solution The computer-generated graph shown in Figure 1.12a is incomplete. Since $\frac{x}{|x|}$ is undefined at $x = 0$, there is no point at $x = 0$. The graph in Figure 1.12b correctly shows open circles at the intersections of the two halves of the graph with the y-axis. We also have

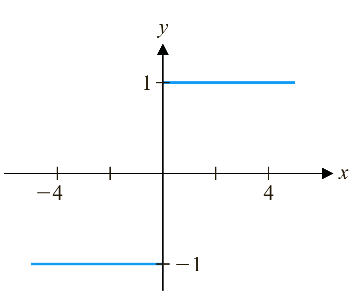


FIGURE 1.12a

$y = \frac{x}{|x|}$

$$\begin{aligned}\lim_{x\to 0^+} \frac{x}{|x|} &= \lim_{x\to 0^+} \frac{x}{x} \qquad \text{Since } |x| = x, \text{ when } x > 0.\\ &= \lim_{x\to 0^+} 1\\ &= 1\end{aligned}$$

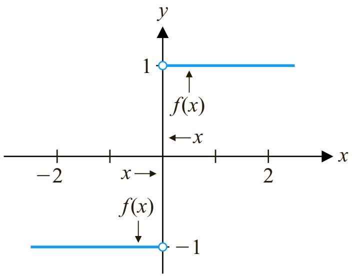


FIGURE 1.12b
$\lim_{x\to 0} \frac{x}{|x|}$ does not exist.

and
$$\lim_{x\to 0^-} \frac{x}{|x|} = \lim_{x\to 0^-} \frac{x}{-x} \qquad \text{Since } |x| = -x, \text{ when } x < 0.$$
$$= \lim_{x\to 0^-} -1$$
$$= -1.$$

It now follows that
$$\lim_{x\to 0} \frac{x}{|x|} \text{ does not exist,}$$

since the one-sided limits are not the same. You should also keep in mind that this observation is entirely consistent with what we see in the graph. ■

EXAMPLE 2.6 A Limit Describing the Movement of a Baseball Pitch

The knuckleball is one of the most exotic pitches in baseball. Batters describe the ball as unpredictably moving left, right, up and down. For a typical knuckleball speed of 60 mph, the left/right position of the ball (in feet) as it crosses the plate is given by

$$f(\omega) = \frac{1.7}{\omega} - \frac{5}{8\omega^2}\sin(2.72\omega)$$

(derived from experimental data in Watts and Bahill's book *Keeping Your Eye on the Ball*), where ω is the rotational speed of the ball in radians per second and where $f(\omega) = 0$ corresponds to the middle of home plate. Folk wisdom among baseball pitchers has it that the less spin on the ball, the better the pitch. To investigate this theory, we consider the limit of $f(\omega)$ as $\omega \to 0^+$. As always, we look at a graph (see Figure 1.13) and generate a table of function values. The graphical and numerical evidence suggests that $\lim_{\omega\to 0^+} f(\omega) = 0$.

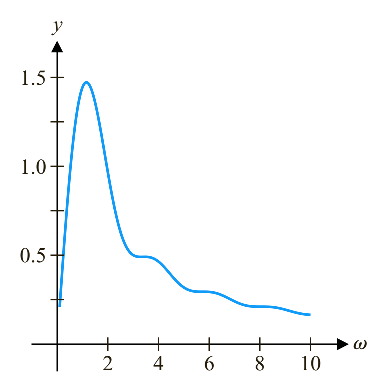


FIGURE 1.13
$y = \frac{1.7}{\omega} - \frac{5}{8\omega^2}\sin(2.72\omega)$

ω	$f(\omega)$
10	0.1645
1	1.4442
0.1	0.2088
0.01	0.021
0.001	0.0021
0.0001	0.0002

The limit indicates that a knuckleball with absolutely no spin doesn't move at all (and therefore would be easy to hit). According to Watts and Bahill, a very slow rotation rate of about 1 to 3 radians per second produces the best pitch (i.e., the most movement). Take another look at Figure 1.13 to convince yourself that this makes sense. ■

EXERCISES 1.2

WRITING EXERCISES

1. Suppose your professor says, "You can think of the limit of $f(x)$ as x approaches a as *what* $f(a)$ *should be*." Critique this statement. What does it mean? Does it provide important insight? Is there anything misleading about it? Replace the phrase in italics with your own best description of what the limit is.

2. Your friend's professor says, "The limit is a *prediction* of what $f(a)$ will be." Compare and contrast this statement to the one in exercise 1. Does the inclusion of the word *prediction* make the limit idea seem more useful and important?

3. We have observed that $\lim_{x\to a} f(x)$ does not depend on the actual value of $f(a)$, or even on whether $f(a)$ exists. In principle, functions such as $f(x) = \begin{cases} x^2 & \text{if } x \neq 2 \\ 13 & \text{if } x = 2 \end{cases}$ are as "normal" as functions such as $g(x) = x^2$. With this in mind, explain why it is important that the limit concept is independent of how (or whether) $f(a)$ is defined.

4. The most common limit encountered in everyday life is the *speed limit*. Describe how this type of limit is very different from the limits discussed in this section.

---

1. For the function graphed below, identify each limit or state that it does not exist.

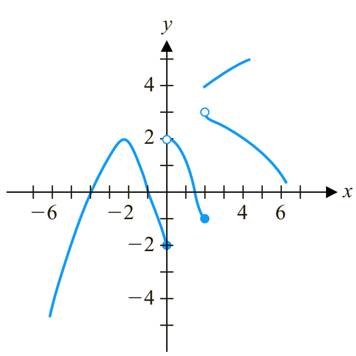
(a) $\lim_{x\to 0^-} f(x)$ (b) $\lim_{x\to 0^+} f(x)$

(c) $\lim_{x\to 0} f(x)$ (d) $\lim_{x\to 1^-} f(x)$

(e) $\lim_{x\to -1} f(x)$ (f) $\lim_{x\to 2^-} f(x)$

(g) $\lim_{x\to 2^+} f(x)$ (h) $\lim_{x\to 2} f(x)$

(i) $\lim_{x\to -2} f(x)$ (j) $\lim_{x\to 3} f(x)$



2. For the function graphed below, identify each limit or state that it does not exist.

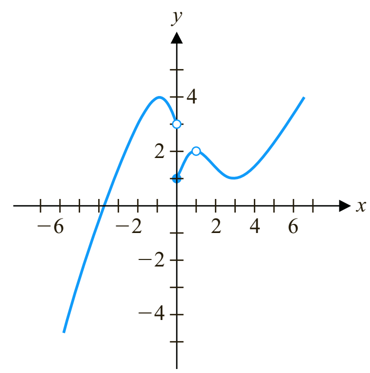
(a) $\lim_{x\to 0^-} f(x)$ (b) $\lim_{x\to 0^+} f(x)$

(c) $\lim_{x\to 0} f(x)$ (d) $\lim_{x\to 2^-} f(x)$

(e) $\lim_{x\to -2} f(x)$ (f) $\lim_{x\to 1^-} f(x)$

(g) $\lim_{x\to 1^+} f(x)$ (h) $\lim_{x\to 1} f(x)$

(i) $\lim_{x\to -1} f(x)$ (j) $\lim_{x\to 3} f(x)$



3. Sketch the graph of $f(x) = \begin{cases} 2x & \text{if } x < 2 \\ x^2 & \text{if } x \ge 2 \end{cases}$ and identify each limit.

(a) $\lim_{x\to 2^-} f(x)$ (b) $\lim_{x\to 2^+} f(x)$

(c) $\lim_{x\to 2} f(x)$ (d) $\lim_{x\to 1} f(x)$

4. Sketch the graph of $f(x) = \begin{cases} x^3 - 1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ \sqrt{x+1} - 2 & \text{if } x > 0 \end{cases}$ and identify each limit.

(a) $\lim_{x\to 0^-} f(x)$ (b) $\lim_{x\to 0^+} f(x)$ (c) $\lim_{x\to 0} f(x)$

(d) $\lim_{x\to -1} f(x)$ (e) $\lim_{x\to 3} f(x)$

5. Sketch the graph of $f(x) = \begin{cases} x^2 + 1 & \text{if } x < -1 \\ 3x + 1 & \text{if } x \ge -1 \end{cases}$ and identify each limit.

(a) $\lim_{x\to -1^-} f(x)$ (b) $\lim_{x\to -1^+} f(x)$

(c) $\lim_{x\to -1} f(x)$ (d) $\lim_{x\to 1} f(x)$

6. Sketch the graph of $f(x) = \begin{cases} 2x+1 & \text{if } x < -1 \\ 3 & \text{if } -1 \le x < 1 \\ 2x+1 & \text{if } x > 1 \end{cases}$ and identify each limit.

(a) $\lim_{x\to -1^-} f(x)$ (b) $\lim_{x\to -1^+} f(x)$ (c) $\lim_{x\to -1} f(x)$
(d) $\lim_{x\to 1} f(x)$ (e) $\lim_{x\to 0} f(x)$

7. Evaluate $f(1.5)$, $f(1.1)$, $f(1.01)$ and $f(1.001)$, and conjecture a value for $\lim_{x\to 1^+} f(x)$ for $f(x) = \dfrac{x-1}{\sqrt{x}-1}$. Evaluate $f(0.5)$, $f(0.9)$, $f(0.99)$ and $f(0.999)$, and conjecture a value for $\lim_{x\to 1^-} f(x)$ for $f(x) = \dfrac{x-1}{\sqrt{x}-1}$. Does $\lim_{x\to 1} f(x)$ exist?

8. Evaluate $f(-1.5)$, $f(-1.1)$, $f(-1.01)$ and $f(-1.001)$, and conjecture a value for $\lim_{x\to -1^-} f(x)$ for $f(x) = \dfrac{x+1}{x^2-1}$. Evaluate $f(-0.5)$, $f(-0.9)$, $f(-0.99)$ and $f(-0.999)$, and conjecture a value for $\lim_{x\to -1^+} f(x)$ for $f(x) = \dfrac{x+1}{x^2-1}$. Does $\lim_{x\to -1} f(x)$ exist?

In exercises 9–14, use numerical and graphical evidence to conjecture values for each limit.

9. $\lim_{x\to 1} \dfrac{x^2-1}{x-1}$

10. $\lim_{x\to -1} \dfrac{x^2+x}{x^2-x-2}$

11. $\lim_{x\to 0} \dfrac{x^2+x}{\sin x}$

12. $\lim_{x\to \pi} \dfrac{\sin x}{x-\pi}$

13. $\lim_{x\to 0} \dfrac{\tan x}{\sin x}$

14. $\lim_{x\to 0} e^{-1/x^2}$

In exercises 15–26, use numerical and graphical evidence to conjecture whether the limit at $x = a$ exists. If not, describe what is happening at $x = a$ graphically.

15. $\lim_{x\to 1} \dfrac{x^2-1}{x^2-2x+1}$

16. $\lim_{x\to -1} \dfrac{x^2-1}{x+1}$

17. $\lim_{x\to 1} \dfrac{\sqrt{5-x}-2}{\sqrt{10-x}-3}$

18. $\lim_{x\to 0} \dfrac{x^2+4x}{\sqrt{x^3+x^2}}$

19. $\lim_{x\to 0} \sin\left(\dfrac{1}{x}\right)$

20. $\lim_{x\to 0} x \sin\left(\dfrac{1}{x}\right)$

21. $\lim_{x\to 2} \dfrac{x-2}{|x-2|}$

22. $\lim_{x\to -1} \dfrac{|x+1|}{x^2-1}$

23. $\lim_{x\to 0} \ln x$

24. $\lim_{x\to 0} x \ln (x^2)$

25. $\lim_{x\to 0} \dfrac{\tan x}{x}$

26. $\lim_{x\to 0} \dfrac{\tan^{-1} x}{x}$

27. Compute $\lim_{x\to 1} \dfrac{x^2+1}{x-1}$, $\lim_{x\to 2} \dfrac{x+1}{x^2-4}$ and similar limits to investigate the following. Suppose that $f(x)$ and $g(x)$ are polynomials with $g(a) = 0$ and $f(a) \neq 0$. What can you conjecture about $\lim_{x\to a} \dfrac{f(x)}{g(x)}$?

28. Compute $\lim_{x\to -1} \dfrac{x+1}{x^2+1}$, $\lim_{x\to \pi} \dfrac{\sin x}{x}$ and similar limits to investigate the following. Suppose that $f(x)$ and $g(x)$ are functions with $f(a) = 0$ and $g(a) \neq 0$. What can you conjecture about $\lim_{x\to a} \dfrac{f(x)}{g(x)}$?

In exercises 29–32, sketch a graph of a function with the given properties.

29. $f(-1) = 2$, $f(0) = -1$, $f(1) = 3$ and $\lim_{x\to 1} f(x)$ does not exist.

30. $f(x) = 1$ for $-2 \le x \le 1$, $\lim_{x\to 1^+} f(x) = 3$ and $\lim_{x\to -2} f(x) = 1$.

31. $f(0) = 1$, $\lim_{x\to 0^-} f(x) = 2$ and $\lim_{x\to 0^+} f(x) = 3$.

32. $\lim_{x\to 0} f(x) = -2$, $f(0) = 1$, $f(2) = 3$ and $\lim_{x\to 2} f(x)$ does not exist.

33. As we see in Chapter 2, the slope of the tangent line to the curve $y = \sqrt{x}$ at $x = 1$ is given by $m = \lim_{h\to 0} \dfrac{\sqrt{1+h}-1}{h}$. Estimate the slope m. Graph $y = \sqrt{x}$ and the line with slope m through the point $(1, 1)$.

34. As we see in Chapter 2, the velocity of an object that has traveled $\sqrt{x}$ miles in x hours at the $x = 1$ hour mark is given by $v = \lim_{x\to 1} \dfrac{\sqrt{x}-1}{x-1}$. Estimate this limit.

35. Consider the following arguments concerning $\lim_{x\to 0^+} \sin \dfrac{\pi}{x}$. First, as $x > 0$ approaches 0, $\dfrac{\pi}{x}$ increases without bound; since $\sin t$ oscillates for increasing t, the limit does not exist. Second: taking $x = 1, 0.1, 0.01$ and so on, we compute $\sin \pi = \sin 10\pi = \sin 100\pi = \cdots = 0$; therefore the limit equals 0. Which argument sounds better to you? Explain. Explore the limit and determine which answer is correct.

36. Consider the following argument concerning $\lim_{x\to 0} e^{-1/x}$. As x approaches 0, $\dfrac{1}{x}$ increases without bound and $\dfrac{-1}{x}$ decreases without bound. Since e^t approaches 0 as t decreases without bound, the limit equals 0. Discuss all the errors made in this argument.

37. Numerically estimate $\lim_{x\to 0^+} (1+x)^{1/x}$ and $\lim_{x\to 0^-} (1+x)^{1/x}$. Note that the function values for $x > 0$ increase as x decreases, while for $x < 0$ the function values decrease as x increases. Explain why this indicates that, if $\lim_{x\to 0} (1+x)^{1/x}$ exists, it is between function values for positive and negative x's. Approximate this limit correct to eight digits.

38. Explain what is wrong with the following logic (you may use exercise 37 to convince yourself that the answer is wrong, but discuss the logic without referring to exercise 37): as $x \to 0$, it is clear that $(1+x) \to 1$. Since 1 raised to any power is 1, $\lim_{x\to 0} (1+x)^{1/x} = \lim_{x\to 0} (1)^{1/x} = 1$.

39. Numerically estimate $\lim_{x\to 0^+} x^{\sec x}$. Try to numerically estimate $\lim_{x\to 0^-} x^{\sec x}$. If your computer has difficulty evaluating the function for negative x's, explain why.

40. Explain what is wrong with the following logic (note from exercise 39 that the answer is accidentally correct): since 0 to any power is 0, $\lim_{x\to 0} x^{\sec x} = \lim_{x\to 0} 0^{\sec x} = 0$.

41. Give an example of a function f such that $\lim_{x\to 0} f(x)$ exists but $f(0)$ does not exist. Give an example of a function g such that $g(0)$ exists but $\lim_{x\to 0} g(x)$ does not exist.

42. Give an example of a function f such that $\lim_{x\to 0} f(x)$ exists and $f(0)$ exists, but $\lim_{x\to 0} f(x) \neq f(0)$.

43. In the text, we described $\lim_{x\to a} f(x) = L$ as meaning "as x gets closer and closer to a, $f(x)$ is getting closer and closer to L." As x gets closer and closer to 0, it is true that x^2 gets closer and closer to -0.01, but it is certainly not true that $\lim_{x\to 0} x^2 = -0.01$. Try to modify the description of limit to make it clear that $\lim_{x\to 0} x^2 \neq -0.01$. We explore a very precise definition of limit in section 1.6.

44. In Figure 1.13, the final position of the knuckleball at time $t = 0.68$ is shown as a function of the rotation rate ω. The batter must decide at time $t = 0.4$ whether to swing at the pitch. At $t = 0.4$, the left/right position of the ball is given by $h(\omega) = \frac{1}{\omega} - \frac{5}{8\omega^2}\sin(1.6\omega)$. Graph $h(\omega)$ and compare to Figure 1.13. Conjecture the limit of $h(\omega)$ as $\omega \to 0$. For $\omega = 0$, is there any difference in ball position between what the batter sees at $t = 0.4$ and what he tries to hit at $t = 0.68$?

45. A parking lot charges \$2 for each hour or portion of an hour, with a maximum charge of \$12 for all day. If $f(t)$ equals the total parking bill for t hours, sketch a graph of $y = f(t)$ for $0 \le t \le 24$. Determine the limits $\lim_{t\to 3.5} f(t)$ and $\lim_{t\to 4} f(t)$, if they exist.

46. For the parking lot in exercise 45, determine all values of a with $0 \le a \le 24$ such that $\lim_{t\to a} f(t)$ does not exist. Briefly discuss the effect this has on your parking strategy (e.g., are there times where you would be in a hurry to move your car or times where it doesn't matter whether you move your car?).

EXPLORATORY EXERCISES

1. In a situation similar to that of example 2.6, the left/right position of a knuckleball pitch in baseball can be modeled by $P = \frac{5}{8\omega^2}(1 - \cos 4\omega t)$, where t is time measured in seconds $(0 \le t \le 0.68)$ and ω is the rotation rate of the ball measured in radians per second. In example 2.6, we chose a specific t-value and evaluated the limit as $\omega \to 0$. While this gives us some information about which rotation rates produce hard-to-hit pitches, a clearer picture emerges if we look at P over its entire domain. Set $\omega = 10$ and graph the resulting function $\frac{1}{160}(1 - \cos 40t)$ for $0 \le t \le 0.68$. Imagine looking at a pitcher from above and try to visualize a baseball starting at the pitcher's hand at $t = 0$ and finally reaching the batter, at $t = 0.68$. Repeat this with $\omega = 5$, $\omega = 1$, $\omega = 0.1$ and whatever values of ω you think would be interesting. Which values of ω produce hard-to-hit pitches?

2. In this exercise, the results you get will depend on the accuracy of your computer or calculator. Work this exercise and compare your results with your classmates' results. We will investigate $\lim_{x\to 0} \frac{\cos x - 1}{x^2}$. Start with the calculations presented in the table (your results may vary):

x	$f(x)$
0.1	$-0.499583\ldots$
0.01	$-0.49999583\ldots$
0.001	$-0.4999999583\ldots$

Describe as precisely as possible the pattern shown here. What would you predict for $f(0.0001)$? $f(0.00001)$? Does your computer or calculator give you this answer? If you continue trying powers of 0.1 (0.000001, 0.0000001 etc.) you should eventually be given a displayed result of -0.5. Do you think this is exactly correct or has the answer just been rounded off? Why is rounding off inescapable? It turns out that -0.5 is the exact value for the limit, so the round-off here is somewhat helpful. However, if you keep evaluating the function at smaller and smaller values of x, you will eventually see a reported function value of 0. This round-off error is not so benign; we discuss this error in section 1.7. For now, evaluate $\cos x$ at the current value of x and try to explain where the 0 came from.

1.3 COMPUTATION OF LIMITS

Now that you have an idea of what a limit is, we need to develop some means of calculating limits of simple functions. In this section, we present some basic rules for dealing with common limit problems. We begin with two simple limits.